Q.1 a. Solve the recurrence relation $T(n) = 27\,T(n/3) + \Theta(n^3 \lg n)$

Answer:

$n^{\log_3 27} = n^3$ vs. $n^3 \lg n$

Therefore $T(n) = O(n^3 \lg_2 n)$

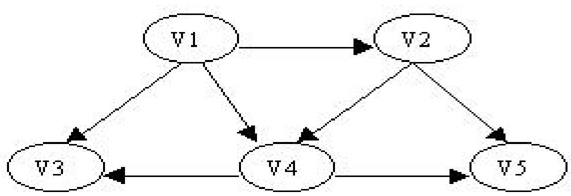
b. Given the following code fragment, what is its Big-O running time?

```
i = n;
while i > 0
  k = k + 2;
 i = i / 2;
```

Answer:

O(log n)

c. Show the ordering of vertices produced by topological sort in the following graph. What is time complexity of topological sort?



Answer:

Topological sort - Order of vertices: V1, V2, V4, V3, V5 or V1, V2, V4, V5, V3

Time complexity: $\Theta(V + E)$

d. Given a sorted array and a value x. Suggest O(n) algorithm to find two values in the array whose sum is equal to x.

Answer:

We keep two indexes one at start and 2nd one at end, and apply following algo. Let the array be sorted in descending order.

```
if(A[1st_index] + A[2nd_index] < x)
   2nd_index--;
else if (A[1st_index] + A[2nd_index] > x)
   1st_index++;
else
    print 1st_index,2nd_index;
do this until 2nd_index > 1st_index
```

e. Suppose that the root of the Red-Black tree is red. If we make it black, does the tree remain a Red Black tree?

Answer:

If we color the root of a relaxed red-black tree black but make no other changes, the resulting tree is a red-black tree. Not even any black-heights change.

f. What are the conditions for a problem to be solved using Dynamic Programming.

Answer:

Optimal substructure and Overlapping sub problems

g. Explain intractable problem with an example.

Answer:

Some problems are intractable as they grow large; we are unable to solve them in reasonable time. e.g. subset-sum problem, TSP etc.

Q.2

a. Give an efficient algorithm that determines whether or not a given directed graph G = (V, E) contains a cycle. Discuss its time complexity.

Answer:

Function iscycle(G)
NV=0; // NV is number of vertices visited
select a vertex that has in degree zero
NV = NV + 1
delete the vertex and all the edges emanating from it from the graph
if NV ≠ V[G] then return “ cycle is there”
else return “ no cycle is there”

Time complexity: In case of Adjacency Matrix $O(V^2)$.
In case of Adjacency List O(V + E).

b. Suppose we wish to search a linked list of length *n*, where each element contains a key *k* along with a hash value *h*(*k*). Each key is a long character string. How might we take advantage of the hash values when searching the list for an element with a given key?

Answer:

Searching a list of length *n* where each element contains a long key *k* and a small hash value *h*(*k*) can be optimized in the following way: Comparing the keys should be done first comparing the hash values and if successful then comparing the keys.

Q.3 a. What is the difference between the binary-search tree property and the heap property? Can the heap property be used to print out the keys of an *n*-node tree in sorted order in O(*n*) time? Explain how or why not.

Answer:

In a heap, a node’s key is ≥ both of its children’s keys. In a binary search tree, a node’s key is ≥ its left child’s key, but ≤ its right child’s key. The heap property, unlike the binary-searth-tree property, doesn’t help print the nodes in sorted order because it doesn’t tell which subtree of a node contains the element to print before that node. In a heap, the largest element smaller than the node could be in either subtree.

Note that if the heap property could be used to print the keys in sorted order in $O(n)$ time, we would have an $O(n)$-time algorithm for sorting, because building the heap takes only $O(n)$ time. But we know that a comparison sort must take $\Omega(n \lg n)$ time.

b. Consider a B-tree with degree *m*. i.e. the number of children *c*, of any internal node (except the root) is such that $m-1 \le c \le 2m-1$. Derive the maximum and minimum number of records in the leaf nodes for such a B-tree with height *h* ($h \ge 1$). (Assume that the root of a tree is at height 0).

Answer:

The root which is at height 0 can have minimum two children. Each of these children can have minimum of m children each of which can have a minimum of m children. Thus the minimum number of records in leaf nodes with height h is $2\times m^{h-1}$.

Similarly the maximum number of records in leaf nodes with height h is $2(2m-1)^{h-1}$.

Q.5

a. Consider the problem of "Making Change". Coins available are:

- **dollars (100 cents)**
- **quarters (25 cents)**
- **dimes (10 cents)**
- **nickels (5 cents)**
- **pennies (1 cent)**

Design an algorithm using greedy approach to make a change of a given amount using the smallest possible number of coins.

Answer:

Informal Algorithm

- Start with nothing.
- at every stage without passing the given amount.
    - add the largest to the coins already chosen.

Formal Algorithm

Make change for n units using the least possible number of coins.

MAKE-CHANGE (n)
    C ← {100, 25, 10, 5, 1}   // constant.
    Sol ← {};   // set that will hold the solution set.
    Sum ← 0 sum of item in solution set
    WHILE sum not = n
        x = largest item in set C such that sum + x ≤ n
        IF no such item **THEN**
            RETURN   "No Solution"
        S ← S {value of x}
        sum ← sum + x
    RETURN S

b. Write a program to merge two arrays in sorted order, so that if an integer is in both the arrays it gets added into the final array only once.

Answer:

Algorithm Union(arr1[], arr2[]):
For union of two arrays, follow the following merge procedure.
1) Use two index variables i and j, initial values i = 0, j = 0
2) If arr1[i] is smaller than arr2[j] then print arr1[i] and increment i.
3) If arr1[i] is greater than arr2[j] then print arr2[j] and increment j.
4) If both are same then print any of them and increment both i and j.
5) Print remaining elements of the larger array.

Q.6 a. How can the output of the Floyd-Warshall algorithm be used to detect the presence of a negative-weight cycle?

Answer:

Here are two ways to detect negative-weight cycles:

(a) Check the main-diagonal entries of the result matrix for a negative value. There is a negative weight cycle if and only if $d_{ii}^{(n)} < 0$ for some vertex i :
- $d_{ii}^{(n)}$ is a path weight from i to itself; so if it is negative, there is a path from i to itself (i.e., a cycle), with negative weight.
- If there is a negative-weight cycle, consider the one with the fewest vertices.
- If it has just one vertex, then some $w_{ii} < 0$, so d_{ii} starts out negative, and since d values are never increased, it is also negative when the algorithm terminates.
- If it has at least two vertices, let k be the highest-numbered vertex in the cycle, and let i be some other vertex in the cycle. $d_{ik}^{(k-1)}$ and $d_{ki}^{(k-1)}$ have correct shortest-path weights, because they are not based on negative weight cycles. (Neither $d_{ik}^{(k-1)}$ nor $d_{ki}^{(k-1)}$ can include k as an intermediate vertex, and i and k are on the negative-weight cycle with the fewest vertices.) Since $i \rightarrow k \rightarrow i$ is a negative-weight cycle, the sum of those two weights is negative, so $d_{ii}^{(k)}$ will be set to a negative value.

Since d values are never increased, it is also negative when the algorithm terminates.

In fact, it suffices to check whether $d_{ii}^{(n-1)} < 0$ for some vertex i . Here's why. A negative-weight cycle containing vertex i either contains vertex n or it does not. If it does not, then clearly $d_{ii}^{(n-1)} < 0$. If the negative-weight cycle contains vertex n, then consider $d_{nn}^{(n-1)}$. This value must be negative, since the cycle, starting and ending at vertex n, does not include vertex n as an intermediate vertex.

(b) Alternatively, one could just run the normal FLOYD-WARSHALL algorithm one extra iteration to see if any of the d values change. If there are negative cycles, then some shortest-path cost will be cheaper. If there are no such cycles, then no d values will change because the algorithm gives the correct shortest paths.

Text Book

Introduction to algorithms- T.M. Cormen, C.E. Leiserson, R.L. Stein, MIT Press, 3rd Edition, 2009